

GENERIC SETS IN SPACES OF MEASURES AND GENERIC SINGULAR CONTINUOUS SPECTRUM FOR DELONE HAMILTONIANS

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ABSTRACT. We show that geometric disorder leads to purely singular continuous spectrum generically.

The main input is a result of Simon known as the “Wonderland theorem”. Here, we provide an alternative approach and actually a slight strengthening by showing that various sets of measures defined by regularity properties are generic in the set of all measures on a locally compact metric space.

As a byproduct we obtain that a generic measure on euclidean space is singular continuous.

1. INTRODUCTION

In this paper we study the spectral type of continuum Hamiltonians describing solids with a specific form of disorder that could be called geometric disorder. The corresponding Hamiltonians are defined using Delone sets (uniformly discrete and uniformly dense subsets of euclidean space) in the following way: the ions of a solid are assumed to be distributed in space according to points of a Delone set. We fix an effective potential v for each of the ions and consider the effective Hamiltonian

$$H(\omega) := -\Delta + \sum_{x \in \omega} v(\cdot - x)$$

for every Delone set $\omega \in \mathbb{D}_{r,R}$ (see below for precise definition). We can now show that under some mild assumptions concerning v, r, R there exists a dense G_δ -set of ω 's for which $H(\omega)$ exhibits a purely singular continuous spectral component.

Let us put this result into perspective. Two extremal forms of geometrically disordered sets are periodic sets and highly random sets with points distributed according to, say, a Poisson law. In the periodic case, it is known that the spectrum is purely absolutely continuous. For the highly random case some pure point spectrum is expected. The region in between these two extremes contains the case of aperiodic order. This form of disorder has attracted much attention recently due to the discovery of solids exhibiting this kind of order (see e.g. [13, 1] for reviews and further references). These solids are called quasicrystals. For them singularly continuous spectrum is assumed to occur.

This picture is well substantiated by one-dimensional investigations [5]. However, in the higher dimensional case essentially nothing is known (except of course absolute continuity of the spectrum for the periodic case).

Thus, our result provides a first modest step in the investigation of geometric disorder in the higher dimensional case.

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The main tool we use goes back to Simon's Wonderland theorem. It basically tells us that certain sets of operators defined by spectral types are in fact regular in the sense that they are G_δ -sets. If, by chance, one can prove denseness of these sets, then their intersection is dense as well. For instance, in the situation indicated above, we can prove that a dense set of ω 's leads to continuous spectral measures (actually, even absolutely continuous spectral measures) and a dense set of ω 's leads to singular spectral measures (in some energy interval). Since both these sets turn out to be G_δ , we get a dense G_δ -set of ω 's such that the spectrum of $H(\omega)$ is singular continuous.

We need a slight generalization of Simon's results. More importantly, we provide a new proof of his result which gives some new insights. Namely, we consider the relevant sets directly in spaces of measures and show that the singular (w.r.t. some comparison measure) measures form a G_δ -set in the space of Borel measures on a nice metric space. The same holds for the continuous measures. By standard continuity arguments, this regularity can be pulled back to spaces of operators.

Since we work directly in spaces of measures we obtain as a byproduct of our investigation that the set of singular continuous measures on euclidean space is a dense G_δ -set in the space of measures. Although this could certainly be expected in view of all the strange things that typically happen as a consequence of Baire's theorem we are not aware of a proof of this fact.

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2. BACK TO WONDERLAND

The following result is the main abstract input to our proof of generic appearance of a purely singular continuous component in the spectrum of certain Delone Hamiltonians.

It is a soft result that basically follows from Baire's Theorem and only a minor strengthening of Simon's "Wonderland Theorem" from [15]. In order to formulate it efficiently, let us introduce the following notation:

For a fixed separable Hilbert space \mathfrak{H} consider the space $\mathfrak{S} = \mathfrak{S}(\mathfrak{H})$ of self-adjoint operators in \mathfrak{H} . We endow \mathfrak{S} with the *strong resolvent topology* τ_{srs} , the weakest topology for which all the mappings

$$\mathfrak{S} \rightarrow \mathbb{C}, A \mapsto (A + i)^{-1}\xi \quad (\xi \in \mathfrak{H})$$

are continuous. Therefore, a sequence (A_n) converges to A w.r.t. τ_{srs} if and only if

$$(A_n + i)^{-1}\xi \rightarrow (A + i)^{-1}\xi$$

for all $\xi \in \mathfrak{H}$.

We denote by $\sigma_*(A)$ for $* = c, s, ac, sc, pp$ the continuous, singular, absolutely continuous, singular continuous and pure point spectrum, respectively.

2.1. Theorem. *Let (X, ρ) be a complete metric space and $H : (X, \rho) \rightarrow (\mathfrak{S}, \tau_{srs})$ a continuous mapping. Assume that, for an open set $U \subset \mathbb{R}$,*

- (1) *the set $\{x \in X \mid \sigma_{pp}(H(x)) \cap U = \emptyset\}$ is dense in X ,*
- (2) *the set $\{x \in X \mid \sigma_{ac}(H(x)) \cap U = \emptyset\}$ is dense in X ,*
- (3) *the set $\{x \in X \mid U \subset \sigma(H(x))\}$ is dense in X .*

Then, the set

$$\{x \in X \mid U \subset \sigma(H(x)), \sigma_{ac}(H(x)) \cap U = \emptyset, \sigma_{pp}(H(x)) \cap U = \emptyset\}$$

is a dense G_δ -set in X .

Remark. If the mapping H in the above theorem happens to be injective, then its range $H(X)$ can be endowed with the metric ρ given on X . In this case, $H(X)$ is a regular metric space of operators in the sense of [15] and Theorem 2.1 of the latter paper gives the result.

As can be seen from this remark, the above theorem is only slightly stronger than Simon's result. Our proof, however, is somewhat different. It is based on the fact that certain sets in spaces of measures are G_δ 's. In that sense we get some extra information in comparison with [15]. The basic idea is that once one has established the G_δ -property for the set of all measures that are purely continuous and purely singular respectively, this regularity can be pulled back to obtain the G_δ -property for the sets appearing in assumptions (1) and (2) of the theorem. Clearly this gives the asserted denseness of the intersection of these sets. Since the set from (3) is G_δ as well, the set appearing in the assertion is the intersection of three dense G_δ -sets and dense as such.

We now start our program of proof by studying certain subsets of the set of positive, regular Borel measures $\mathcal{M}_+(S)$ on some locally compact, σ -compact, separable metric space S . Of course, $\mathcal{M}_+(S)$ is endowed with the weak topology from $C_c(S)$, also called the vague topology. We refer the reader to [3] for standard results concerning the space of measures. In particular, we note that the vague topology is metrizable such that $\mathcal{M}_+(S)$ becomes a complete metric space, as we consider second countable spaces. For the application we have in mind, S is just the open subset U of the real line that appears in the theorem.

We call a measure $\mu \in \mathcal{M}_+(S)$ *diffusive* or *continuous* if its *atomic* or *pure point* part vanishes, i.e. if $\mu(\{x\}) = 0$ for every $x \in S$. (We prefer the former terminology in the abstract framework and the latter for measures on the real line.) Two measures are said to be *mutually singular*, $\mu \perp \nu$, if there exists a set $C \subset S$ such that $\mu(C) = 0 = \nu(S \setminus C)$. We have

2.2. Theorem. *Let S be as above.*

- (1) *The set $\{\mu \in \mathcal{M}_+(S) \mid \mu$ is diffusive $\}$ is a G_δ -set in $\mathcal{M}_+(S)$.*
- (2) *For any $\lambda \in \mathcal{M}_+(S)$, the set $\{\mu \in \mathcal{M}_+(S) \mid \mu \perp \lambda\}$ is a G_δ -set in $\mathcal{M}_+(S)$.*
- (3) *For any closed $F \subset S$ the set $\{\mu \in \mathcal{M}_+(S) \mid F \subset \text{supp}(\mu)\}$ is a G_δ -set in $\mathcal{M}_+(S)$.*

Proof of Theorem 2.2 (3): First consider the case that $F = \{x\}$. Choose a basis $(V_n)_{n \in \mathbb{N}}$ of open neighborhoods of x . Then

$$\{\mu \in \mathcal{M}_+(S) \mid x \in \text{supp}(\mu)\}^c = \bigcup_{n \in \mathbb{N}} \{\mu \in \mathcal{M}_+(S) \mid \mu(V_n) = 0\}$$

is an F_σ , since $\{\mu \in \mathcal{M}_+(S) \mid \mu(V) = 0\}$ is closed for any open set $V \subset S$. Therefore $\{\mu \in \mathcal{M}_+(S) \mid x \in \text{supp}(\mu)\}$ is a G_δ .

To treat the general case, take a dense subset $\{x_n \mid n \in \mathbb{N}\}$ in F (which is possible since S is separable.) Then

$$\{\mu \in \mathcal{M}_+(S) \mid F \subset \text{supp}(\mu)\} = \bigcap_{n \in \mathbb{N}} \{\mu \in \mathcal{M}_+(S) \mid x_n \in \text{supp}(\mu)\}$$

is a countable intersection of G_δ 's, hence a G_δ . □

To prove parts (1) and (2) of Theorem 2.2 we use the following observations:

2.3. Proposition. *Let $\mathcal{K} \subset \mathcal{M}_+(S)$ be compact. Then*

$$\mathcal{K}^\bullet := \{\mu \in \mathcal{M}_+(S) \mid \exists \nu \in \mathcal{K} : \nu \leq \mu\}$$

is closed.

Proof. Let (μ_n) be a sequence in \mathcal{K}^\bullet that converges to μ . Choose $\nu_n \in \mathcal{K}$ with $\nu_n \leq \mu_n$ for $n \in \mathbb{N}$. By compactness, we find a converging subsequence (ν_{n_k}) with limit $\nu \in \mathcal{K}$. For any $\varphi \in C_c(S), \varphi \geq 0$, we get

$$\begin{aligned} \langle \mu, \varphi \rangle &= \lim_{k \rightarrow \infty} \langle \mu_{n_k}, \varphi \rangle \\ &\geq \lim_{k \rightarrow \infty} \langle \nu_{n_k}, \varphi \rangle \\ &= \langle \nu, \varphi \rangle \end{aligned}$$

so that $\mu \in \mathcal{K}^\bullet$. □

2.4. Proposition. *Let $K \subset S$ be compact and $a > 0$. Then*

$$\{a \cdot \delta_x \mid x \in K\}$$

is compact in $\mathcal{M}_+(S)$.

The *proof* is evident from the fact that $S \rightarrow \mathcal{M}_+(S), x \mapsto \delta_x$ is continuous.

2.5. Proposition. *Let $\lambda \in \mathcal{M}_+(S)$, $K \subset S$ compact and $\gamma > 0$ be given. Then*

$$\mathcal{K} := \{f \cdot \lambda \mid f \in L^2(\lambda), \|f\|_{L^2(\lambda)} \leq 1, 0 \leq f, \text{supp}(f) \subset K, \int f d\lambda \geq \gamma\}$$

is compact.

Proof. The densities considered in \mathcal{K} form a closed subset of the unit ball of $L^2(K, \lambda)$. Since the latter is weakly compact and the mapping $L^2(K, \lambda)_+ \rightarrow \mathcal{M}_+(S), f \mapsto f\lambda$ is w-w* -continuous we get the desired compactness. □

Proof of Theorem 2.2 (1) and (2): For the proof of (1) consider

$$\mathcal{M}_1 := \{\mu \in \mathcal{M}_+(S) \mid \mu \text{ is diffusive}\}^c.$$

We want to show that \mathcal{M}_1 is an F_σ . By assumption on S , we find a sequence of compacts $K_n \nearrow S$ and get that

$$\mathcal{K}_{1,n} = \left\{ \frac{1}{n} \cdot \delta_x \mid x \in K_n \right\}$$

is compact by Proposition 2.4. Proposition 2.3 yields that $\mathcal{F}_{1,n} = \mathcal{K}_{1,n}^\bullet$ is closed. Since

$$\mathcal{M}_1 = \bigcup_{n \in \mathbb{N}} \mathcal{F}_{1,n}$$

we arrive at the desired conclusion.

We show (2) with a similar argument. For K_n defined as in the proof of (1), let

$$\mathcal{K}_{2,n} = \{f\lambda \mid f \in L^2(\lambda), \|f\| \leq 1, 0 \leq f, \text{supp}(f) \subset K_n, \int f d\lambda \geq \frac{1}{n}\}$$

which is compact by Proposition 2.5. Again by Proposition 2.3 we get that $\mathcal{F}_{2,n} = \mathcal{K}_{2,n}^\bullet$ is closed. Since

$$\mathcal{M}_2 = \bigcup_{n \in \mathbb{N}} \mathcal{F}_{2,n}$$

is an F_σ and

$$\mathcal{M}_2^c = \{\mu \in \mathcal{M}_+(S) \mid \mu \perp \lambda\},$$

part (2) of Theorem 2.2 is proven. \square

We now pull back the regularity properties derived in Theorem 2.2 to regularity properties in \mathfrak{S} . We denote the spectral measure of $A \in \mathfrak{S}$ for $\xi \in \mathfrak{H}$ by ρ_ξ^A . It is defined by

$$\langle \rho_\xi^A, \varphi \rangle = (\varphi(A)\xi | \xi) \text{ for } \varphi \in C_c(\mathbb{R}).$$

It is easy to see that $A_n \xrightarrow{srs} A$ implies strong convergence $\varphi(A_n) \rightarrow \varphi(A)$ for every $\varphi \in C_c(\mathbb{R})$ (see, e.g., [17], Satz 9.20) which in turn gives weak convergence $\rho_\xi^{A_n} \rightarrow \rho_\xi^A$ for every $\xi \in \mathfrak{H}$. Thus, for each fixed $\xi \in \mathfrak{H}$, the map $\mathfrak{S} \rightarrow \mathcal{M}_+(\mathbb{R})$, $A \mapsto \rho_\xi^A$, is continuous.

The spectral subspaces of A are defined by

$$\begin{aligned} \mathfrak{H}_{ac}(A) &= \{\xi \in \mathfrak{H} \mid \rho_\xi^A \text{ is absolutely continuous}\} \\ \mathfrak{H}_{sc}(A) &= \{\xi \in \mathfrak{H} \mid \rho_\xi^A \text{ is singular continuous}\}, \\ \mathfrak{H}_c(A) &= \{\xi \in \mathfrak{H} \mid \rho_\xi^A \text{ is continuous}\} \\ \mathfrak{H}_{pp}(A) &= \mathfrak{H}_c(A)^\perp, \mathfrak{H}_s(A) = \mathfrak{H}_{ac}(A)^\perp. \end{aligned}$$

These subspaces are closed and invariant under A . $\mathfrak{H}_{pp}(A)$ is the closed linear hull of the eigenvectors of A . Recall that the spectra $\sigma_*(A)$ are just the spectra of A restricted to $\mathfrak{H}_*(A)$. Using Theorem 2.2, we get the following:

2.6. Proposition. *Let $U \subset \mathbb{R}$ be open and $\mathfrak{S} \subset \mathfrak{H}$ a closed subspace. Then*

- (1) $\{A \in \mathfrak{S} \mid \forall \xi \in \mathfrak{S} : \rho_\xi^A|_U \text{ is continuous}\}$ is a G_δ ,
- (2) $\{A \in \mathfrak{S} \mid \forall \xi \in \mathfrak{S} : \rho_\xi^A|_U \text{ is singular}\}$ is a G_δ .

Proof. First, fix $\xi \in \mathfrak{H}$. We use that the mappings $\mathcal{M}_+(\mathbb{R}) \rightarrow \mathcal{M}_+(U)$, $\nu \mapsto \nu|_U$ and $\mathfrak{S} \rightarrow \mathcal{M}_+(\mathbb{R})$, $A \mapsto \rho_\xi^A$ are continuous. Therefore we get that

$$\{A \in \mathfrak{S} \mid \rho_\xi^A|_U \text{ is continuous}\}$$

is a G_δ by Theorem 2.2 (1), since continuous is synonymous to diffusive. In the same way

$$\{A \in \mathfrak{S} \mid \rho_\xi^A|_U \text{ is singular}\}$$

is a G_δ by Theorem 2.2 (2), since singular means singular with respect to the Lebesgue measure. Using the above mentioned fact that the spectral subspaces are closed, for any dense set $\{\xi_n \mid n \in \mathbb{N}\}$ we get

$$\{A \in \mathfrak{S} \mid \forall \xi \in \mathfrak{S} : \rho_\xi^A|_U \text{ is continuous}\} = \bigcap_{n \in \mathbb{N}} \{A \in \mathfrak{S} \mid \rho_{\xi_n}^A|_U \text{ is continuous}\}$$

is a G_δ as well as

$$\{A \in \mathfrak{S} \mid \forall \xi \in \mathfrak{S} : \rho_\xi^A|_U \text{ is singular}\} = \bigcap_{n \in \mathbb{N}} \{A \in \mathfrak{S} \mid \rho_{\xi_n}^A|_U \text{ is singular}\}.$$

\square

For completeness sake let us reproduce the well known fact of lower semi continuity of spectra under strong continuity; see, e.g. [15], Lemma 1.6 or [17], Satz 9.26 (b).

2.7. Proposition. *Let $V \subset \mathbb{R}$ be open. Then*

$$\{A \in \mathfrak{S} \mid \sigma(A) \cap V = \emptyset\}$$

is closed.

Proof.

$$\{A \in \mathfrak{S} \mid \sigma(A) \cap V = \emptyset\} = \bigcap_{\varphi \in C_c(V)} \{A \in \mathfrak{S} \mid \varphi(A) = 0\}$$

is closed by the above mentioned continuity of $A \mapsto \varphi(A)$. \square

We can now put all that together for the

Proof of Theorem 2.1: By continuity of H and Proposition 2.6 applied with $\mathfrak{G} = \mathfrak{H}$ we get that the sets appearing in assumptions (1) and (2) of the theorem are G_δ 's. Choosing a countable base $V_n, n \in \mathbb{N}$ of U , we find that

$$\{A \in \mathfrak{S} \mid U \subset \sigma(A)\}^c = \bigcap_{n \in \mathbb{N}} \{A \in \mathfrak{S} \mid \sigma(A) \cap V_n = \emptyset\}$$

is an F_σ . Thus, invoking again the continuity H , we infer that the set appearing in assumption (3) of the theorem is a G_δ . Therefore, the asserted denseness follows by Baire's theorem, since the set appearing in the assertion is just the intersection of the three dense G_δ 's from (1)-(3). \square

Let us end our excursion to Wonderland by emphasizing the special role of singular continuity witnessed above: Let U be an open subset of \mathbb{R}^d . Then, both

$$\mathcal{M}_{ac}(U) := \{\mu \in \mathcal{M}_+(U) \mid \mu \text{ is absolutely continuous}\}$$

and

$$\mathcal{M}_{pp}(U) := \{\mu \in \mathcal{M}_+(U) \mid \mu \text{ is pure point}\}$$

are dense in $\mathcal{M}_+(U)$ as can be seen by standard arguments. Thus, by Theorem 2.2 (1) and (2), the sets $\mathcal{M}_c(U) := \{\mu \in \mathcal{M}_+(U) \mid \mu \text{ is continuous}\}$ and $\mathcal{M}_s(U) := \{\mu \in \mathcal{M}_+(U) \mid \mu \text{ is singular}\}$ are dense G_δ 's. Therefore,

$$\mathcal{M}_{sc}(U) := \{\mu \in \mathcal{M}_+(U) \mid \mu \text{ is singular continuous}\} = \mathcal{M}_c(U) \cap \mathcal{M}_s(U)$$

is a dense G_δ by Baire's theorem. Then, another application of Baire's theorem, shows that neither $\mathcal{M}_{ac}(U)$ nor $\mathcal{M}_{pp}(U)$ can be a G_δ , as they do not intersect $\mathcal{M}_{sc}(U)$.

Let us state the consequences for the special case of euclidean space.

2.8. Corollary. *Let U be an open subset of \mathbb{R}^d . Then the singular continuous measures $\mathcal{M}_{sc}(U)$ form a dense G_δ in the space $\mathcal{M}_+(U)$ of Borel measures.*

Of course, much more general spaces and reference measures can be treated: the only important restriction is that the measures singular w.r.t the reference measure and the diffusive measures form dense sets.

Once more we find that the silent majority consists of rather strange individuals. In mathematical terms, we owe this fact to Baire and completeness. We refer to [18, 19] for a similar result saying that most continuous monotonic functions on the real line are not differentiable. Let us also mention the classical papers [2, 12] dealing with the lack of differentiability for a typical (in the sense of dense G_δ 's) continuous functions.

3. DELONE SETS AND DELONE HAMILTONIANS

Let us now define the operators for which we want to prove generic singular continuous spectral components. We start by recalling what a *Delone set* is, a notion named after B.N. Delone (Delaunay), [6]. We write $U_r(x)$ and $B_r(x)$ for the open and closed ball in \mathbb{R}^d , respectively. The Euclidean norm on \mathbb{R}^d is denoted by $\|\cdot\|$.

Definition. A set $\omega \subset \mathbb{R}^d$ is called an (r, R) -set if

- $\forall x, y \in \omega, x \neq y : U_r(x) \cap U_r(y) = \emptyset$,
- $\bigcup_{x \in \omega} B_R(x) = \mathbb{R}^d$.

By $\mathbb{D}_{r,R}(\mathbb{R}^d) = \mathbb{D}_{r,R}$ we denote the set of all (r, R) -sets. We say that $\omega \subset \mathbb{R}^d$ is a *Delone set*, if it is an (r, R) -set for some $0 < r \leq R$ so that $\mathbb{D}(\mathbb{R}^d) = \mathbb{D} = \bigcup_{0 < r \leq R} \mathbb{D}_{r,R}(\mathbb{R}^d)$ is the set of all Delone sets.

Delone sets turn out to be quite useful in the description of quasicrystals and more general aperiodic solids; see also [4], where the relation to discrete operators is discussed. In fact, if we regard an infinitely extended solid whose ions are assumed to be fixed, then the positions are naturally distributed according to the points of a Delone set. Fixing an effective potential v for all the ions this leads us to consider the Hamiltonian

$$H(\omega) := -\Delta + \sum_{x \in \omega} v(\cdot - x) \text{ in } \mathbb{R}^d,$$

where $\omega \in \mathbb{D}$. Let us assume, for simplicity that v is bounded, measurable and compactly supported.

In order to apply our analysis above, we need to introduce a suitable topology on \mathbb{D} . This can be done in several ways, cf. [4, 10]. We follow the strategy from the latter paper and refer to it for details (see the discussion in the appendix as well). The *natural topology* defines a compact, completely metrizable topology on the set of all closed subsets of \mathbb{R}^d for which $\mathbb{D}_{r,R}(\mathbb{R}^d)$ is a compact, complete space. Moreover, the map

$$H : \mathbb{D}_{r,R}(\mathbb{R}^d) \longrightarrow \mathfrak{S}(L^2((\mathbb{R}^d)), \omega \mapsto H(\omega)),$$

is continuous. This is a straightforward consequence of the following lemma, which describes convergence w.r.t the natural topology.

3.1. Lemma. *A sequence (ω_n) of Delone sets converges to $\omega \in \mathbb{D}$ in the natural topology if and only if there exists for any $l > 0$ an $L > l$ such that the $\omega_n \cap U_L(0)$ converge to $\omega \cap U_L(0)$ with respect to the Hausdorff distance as $n \rightarrow \infty$.*

The *Proof* of the lemma will be given in the appendix. There, we also discuss further features of the natural topology. We say that a Delone set ρ is *crystallographic* if $Per(\rho) := \{t \in \mathbb{R}^d : \rho = t + \rho\}$ is a lattice.

We are now in position to state the main application of this paper:

3.2. Theorem. *Let $r, R > 0$ with $2r \leq R$ and v be given such that there exist crystallographic $\gamma, \tilde{\gamma} \in \mathbb{D}_{r,R}$ with $\sigma(H(\gamma)) \neq \sigma(H(\tilde{\gamma}))$. Then*

$$U := (\sigma(H(\gamma))^{\circ} \setminus \sigma(H(\tilde{\gamma}))) \cup (\sigma(H(\tilde{\gamma}))^{\circ} \setminus \sigma(H(\gamma)))$$

is nonempty and there exists a dense G_{δ} -set $\Omega_{sc} \subset \mathbb{D}_{r,R}$ such that for every $\omega \in \Omega_{sc}$ the spectrum of $H(\omega)$ contains U and is purely singular continuous in U .

To prove the theorem, we need two results on extension of Delone sets. To state the results we use the following notation. For $S > 0$ we define $Q(S) := [-S, S]^d \subset \mathbb{R}^d$.

3.3. Lemma. *Let $r, R > 0$ with $2r \leq R$ be given. Let $\omega \in \mathbb{D}_{r,R}$ and $S > 0$ be arbitrary. Then, there exists a crystallographic $\rho \in \mathbb{D}_{r,R}$ with $\rho \cap Q(S) = \omega \cap Q(S)$.*

Proof. Let $P : \mathbb{R}^d \longrightarrow \mathbb{R}^d / 2(S + R + r)\mathbb{Z}^d =: \mathbb{T}$ be the canonical projection. Note that the Euclidean norm on \mathbb{R}^d induces a canonical metric e on \mathbb{T} with $e(P(x), P(y)) = \|x - y\|$ whenever $x, y \in \mathbb{R}^d$ are close to each other.

Let $F_0 := P(\omega \cap Q(S+R))$. By assumption on ω , we have $e(p, q) \geq 2r$ for all $p, q \in F_0$ with $p \neq q$. Moreover, $\cup_{p \in F_0} B_e(p, R) \supset P(Q(S)) =: C$, where $B_e(p, R)$ denotes the ball around p with radius R in the metric e .

Adding successively points from $\mathbb{T} \setminus C$ to F_0 we obtain a finite set F which is maximal among the sets satisfying

$$e(p, q) \geq 2r, \text{ for all } p, q \in F \text{ with } p \neq q.$$

As any larger set violates this condition and $R \geq 2r$, we infer

$$\cup_{p \in F} B_e(p, R) = \mathbb{T}.$$

Now, $\rho := P^{-1}(F)$ has the desired properties. \square

3.4. Lemma. *Let $r, R > 0$ with $2r \leq R$ be given. Let $\gamma, \omega \in \mathbb{D}_{r,R}$ and $S > 0$ be arbitrary. Then, there exists a $\rho \in \mathbb{D}_{r,R}$ with*

$$\rho \cap Q(S) = \omega \cap Q(S) \text{ and } \rho \cap (\mathbb{R}^d \setminus Q(S + 2R + r)) = \gamma \cap (\mathbb{R}^d \setminus Q(S + 2R + r)).$$

Proof. Define

$$\rho' := (\omega \cap Q(S + R)) \cup (\gamma \cap (\mathbb{R}^d \setminus Q(S + R + r))).$$

Then, $\cup_{x \in \rho'} B(x, R) \supset Q(S) \cup (\mathbb{R}^d \setminus Q(S + 2R + r))$ and $\|x - y\| \geq 2r$ for all $x, y \in \rho'$ with $x \neq y$. Adding successively points from $Q(S + 2R + r) \setminus Q(S)$ to ρ we arrive at the desired set ρ . \square

Proof of Theorem 3.2. We let $U_1 := \sigma(H(\gamma))^\circ \setminus \sigma(H(\tilde{\gamma}))$ and $U_2 := \sigma(H(\tilde{\gamma}))^\circ \setminus \sigma(H(\gamma))$. Since $\gamma, \tilde{\gamma}$ are crystallographic, the corresponding operators are periodic and their spectra are consequently purely absolutely continuous and consist of a union of closed intervals with only finitely many gaps in every compact subset of the reals. Hence, under the assumption of the theorem U_1 or U_2 is nonempty. Thus, U is nonempty.

We now consider the case that U_1 is nonempty. We will verify conditions (1)-(3) from Theorem 2.1 above:

Ad (1): Fix $\omega \in \mathbb{D}_{r,R}$. For $n \in \mathbb{N}$ consider $\nu_n := \omega \cap Q(n)$. By Lemma 3.3, we can find a crystallographic ω_n in $\mathbb{D}_{r,R}$ with $\omega_n \cap Q(n) = \nu_n$. For given $L > 0$ we get that $\omega_n \cap U_L(0) = \omega \cap U_L(0)$ if n is large enough. Therefore, by Lemma 3.1, we find that $\omega_n \rightarrow \omega$ with respect to the natural topology. On the other hand, $\sigma_{pp}(H(\omega_n)) = \emptyset$ since the potential of $H(\omega_n)$ is periodic. Consequently,

$$\{\omega \in \mathbb{D}_{r,R} \mid \sigma_{pp}(H(\omega)) \cap U_1 = \emptyset\}$$

is dense in $\mathbb{D}_{r,R}$.

Ad (2): To get the denseness of ω for which $\sigma_{ac}(H(\omega)) \cap U_1 = \emptyset$, fix $\omega \in \mathbb{D}_{r,R}$. For $n \in \mathbb{N}$ large enough, we apply Lemma 3.4 to obtain $\omega_n \in \mathbb{D}_{r,R}$ such that

$$\omega_n \cap U_n(0) = \omega \cap U_n(0) \text{ and } \omega_n \cap U_{2n}(0)^c = \tilde{\gamma} \cap U_{2n}(0)^c.$$

In virtue of the last property, $H(\omega_n)$ and $H(\tilde{\gamma})$ only differ by a compactly supported, bounded potential, so that $\sigma_{ac}(H(\omega_n)) = \sigma_{ac}(H(\tilde{\gamma})) \subset U_1^c$. In fact, standard arguments of scattering theory give that $e^{-H(\omega_n)} - e^{-H(\tilde{\gamma})}$ is a trace class operator. By the invariance principle, the wave operators for $H(\omega_n)$ and $H(\tilde{\gamma})$ exist and are complete which

in turn implies equality of the absolutely continuous spectra. See, e.g., [14], Section XI.3 and [7], Section 2 as well as [16], Corollary of Theorem 2 for a much more general result.

Again, $\omega_n \rightarrow \omega$ yields condition (2) of Theorem 2.1.

Ad (3): This can be checked with a similar argument, this time with $\tilde{\gamma}$ instead of γ . More precisely, we proceed as follows: Fix $\omega \in \mathbb{D}_{r,R}$. For $n \in \mathbb{N}$ large enough, we apply Lemma 3.4 to obtain $\omega_n \in \mathbb{D}_{r,R}$ such that

$$\omega_n \cap U_n(0) = \omega \cap U_n(0) \text{ and } \omega_n \cap U_{2n}(0)^c = \gamma \cap U_{2n}(0)^c.$$

In virtue of the last property, $H(\omega_n)$ and $H(\gamma)$ only differ by a compactly supported, bounded potential, so that $\sigma_{ac}(H(\omega_n)) = \sigma_{ac}(H(\gamma)) \supset U_1$. By $\omega_n \rightarrow \omega$, we obtain (3) of Theorem 2.1.

Summarizing what we have shown so far, an appeal to Theorem 2.1 gives that

$$\{\omega \in \mathbb{D}_{r,R} \mid \sigma_{pp}(H(\omega)) \cap U_1 = \emptyset, \sigma_{ac}(H(\omega)) \cap U_1 = \emptyset, U_1 \subset \sigma(H(\omega))\}$$

is a dense G_δ -set if U_1 is not empty.

An analogous argument shows the same statement with U_2 instead of U_1 . This proves the assertion if only one of the U_i , $i = 1, 2$, is not empty. Otherwise, the assertion follows after intersecting the two dense G_δ 's. \square

Remark. The assumption of the above theorem combines non triviality of v and the existence of suitable crystallographic $\gamma, \tilde{\gamma}$.

A simple way of ensuring this condition for $v \neq 0$ of fixed sign is to choose $R > 2r$. By an argument as in the proof of Lemma 3.3 we can then find crystallographic $\gamma, \tilde{\gamma}$ such that $0 \in \tilde{\gamma} \setminus \gamma$. The corresponding periodic operators differ by a periodic potential with the same periodicity lattice. In this case, the analysis of [8] can be applied, showing that the spectra of $H(\tilde{\gamma})$ and $H(\gamma)$ differ.

APPENDIX A. THE NATURAL TOPOLOGY AND THE PROOF OF LEMMA 3.1

The purpose of this section is to comment on the natural topology and to give a proof for Lemma 3.1. To do so we use the description of the natural topology via a one point compactification as given in our article [10].

We need some notation. Whenever (X, e) is a metric space, the Hausdorff distance e_H on the compact subsets of X is defined by

$$e_H(K_1, K_2) := \inf(\{\epsilon > 0 : K_1 \subset U_\epsilon(K_2) \text{ and } K_2 \subset U_\epsilon(K_1)\} \cup \{1\}),$$

where K_1, K_2 are compact subsets of the metric space (X, e) and $U_\epsilon(K)$ denotes the open ϵ -neighborhood around K w.r.t e . It is well known that the set of all compact subsets of X becomes a complete compact metric space in this way whenever (X, e) is complete and compact.

Using the stereographic projection

$$j : \mathbb{R}^d \cup \{\infty\} \longrightarrow \mathbb{S}^d := \{x \in \mathbb{R}^{d+1} : \|x\| = 1\}$$

we can identify $\mathbb{R}^d \cup \{\infty\}$ and \mathbb{S}^d . We denote the Euclidean distance on \mathbb{S}^d by ρ . Denote by $\mathcal{F} := \mathcal{F}(\mathbb{R}^d)$ the set of closed subsets of \mathbb{R}^d . We can then define a metric δ on $\mathcal{F}(\mathbb{R}^d)$ by

$$\delta(F, G) := \rho_H(j(F \cup \{\infty\}), j(G \cup \{\infty\})).$$

This makes sense since $j(F \cup \{\infty\})$ is compact in (\mathbb{S}^d, ρ) whenever F is closed in \mathbb{R}^d . Moreover, it is not hard to see that $\{j(F \cup \{\infty\}) : F \in \mathbb{D}_{r,R}\}$ is closed within the compact subsets of \mathbb{S}^d and hence compact.

After this general discussion, we can now give a

Proof of Lemma 3.1. We start by proving the “only if” part: Let $\omega_n, \omega \in \mathbb{D}$ be given with $\omega_n \rightarrow \omega$ in (\mathcal{F}, δ) . Fix $l > 0$. Since ω is discrete, we find $L > l$ such that $d(\partial U_L(0), \omega) =: \gamma > 0$, where d refers to the Euclidean distance. There clearly exists C_L with

$$\rho(j(x), j(y)) \leq 2\|x - y\| \leq C_L \rho(j(x), j(y)) \quad (*)$$

for all $x, y \in U_L(0)$. Fix $\epsilon > 0$ and use convergence of ω_n to ω in \mathcal{F} to find $n_0 \in \mathbb{N}$ such that $\delta(\omega_n, \omega) \leq \beta\epsilon$ for $n \geq n_0$ for β small enough so that

$$C_L \beta \epsilon \leq \min\{\epsilon, \gamma\}.$$

By definition of δ this means that for every $x \in \omega \cap U_L(0)$ there is $x_n \in \omega_n$ such that $\rho(j(x), j(x_n)) \leq \beta\epsilon$. From $(*)$, we get that

$$\|x - x_n\| \leq \frac{C_L}{2} \beta \epsilon \leq \frac{1}{2} \min\{\epsilon, \gamma\}$$

so that $x_n \in U_L(0)$ and $\omega \cap U_L(0) \subset U_\epsilon(\omega_n \cap U_L(0))$ for all $n \geq n_0$.

Conversely, given $x_n \in \omega_n \cap U_L(0)$ we find $x \in \omega$ such that $\rho(j(x), j(x_n)) \leq \beta\epsilon$. Again, by our choice of γ, β and $j(x) \in U_L(0)$, this implies $\omega_n \cap U_L(0) \subset U_\epsilon(\omega \cap U_L(0))$.

These considerations show that $\omega_n \rightarrow \omega$ in (\mathcal{F}, δ) implies that for all $l > 0$ there exists $L > l$ with $d_H(\omega_n \cap U_L(0), \omega \cap U_L(0)) \rightarrow 0$. This proves the “only if” part of the lemma.

We are now going to prove the “if” part of the lemma. Assume that ω_n, ω are Delone sets satisfying the condition of the lemma. We use a standard compactness argument to show that $\omega_n \rightarrow \omega$ w.r.t. δ . Choose an arbitrary subsequence (ω_{n_k}) of (ω_n) . By compactness, there is a subsequence $(\omega_{n_{k_l}})$ converging to some $\tilde{\omega}$ w.r.t. δ : By the first part of the proof, for every $l > 0$, we then find $L > 0$ with $d_H(\omega_{n_{k_l}} \cap U_L(0), \tilde{\omega} \cap U_L(0)) \rightarrow 0$. This implies that $\omega = \tilde{\omega}$. Therefore, every subsequence of (ω_n) has a subsequence converging to ω w.r.t. δ . Thus, the sequence (ω_n) itself converges to ω w.r.t. δ . This finishes the proof of the lemma. \square

The proof of the lemma shows effectively that $\omega_n \rightarrow \omega$ in (\mathcal{F}, δ) if and only if the following two conditions hold:

- (i) For every $x \in \omega$, there exists (x_n) with $x_n \in \omega_n$ for every $n \in \mathbb{N}$ and $x_n \rightarrow x$.
- (ii) Whenever (x_n) is a sequence with $x_n \in \omega_n$ for every $n \in \mathbb{N}$ and $x_n \rightarrow x$, then $x \in \omega$.

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